Linear Transformations : A Geometric Approach

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Introduction

Linear Transformations

In mathematics, linear transformation is a mapping between two vector spaces that preserves the operations of vector addition and scalar multiplication. A linear Transformations preserves algebraic operations.

- The sum of two vectors is mapped to the sum of their images.
- Scalar multiplication of a vector is mapped to scalar multiple of its images.

Introduction

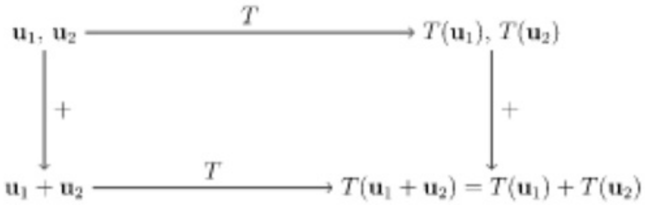
Linear Transformations

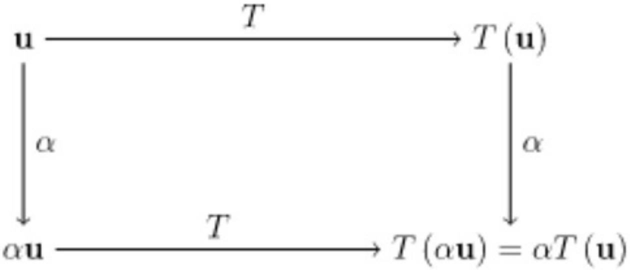
linear transformation is a mapping $T : V \to W$, the following two conditions are satisfied:

- Additivity / operation of addition $\mathrm{T}(\mathrm{u}_1 + \mathrm{u}_2) = \mathrm{T}(\mathrm{u}_1) + \mathrm{T}(\mathrm{u}_2)$
- Operation of scalar multiplication : $\mathrm{T}(\alpha \mathrm{u}) = \alpha \mathrm{T}(\mathrm{u})$

for any two vectors $\mathrm{u}_1, \mathrm{u}_1 \in$V and any scalar $\alpha \in F$.

Thus, a linear map is said to be operation preserving. In other words, it does not matter whether the linear map is applied before (the right hand sides of the above examples) or after (the left hand sides of the examples) the operations of addition and scalar multiplication.


$\mathbf{u}_1, \mathbf{u}_2$
T
$T(\mathbf{u}_1), T(\mathbf{u}_2)$
$+$
$+$
$\mathbf{u}_1 + \mathbf{u}_2$
T
$T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$



u
T
T (u)
α
α
αu
T
T (αu) = αT (u)

Geometric Interpretation

Geometric Interpretation

A linear map sends lines passing through the origin to lines passing through the origin or onto the origin.

Linear Transformations

Properties

A mapping T : V $\rightarrow$ W then

- If T is linear transformation then $T(\theta){=}\theta'$ where θ is null the vector of V and θ' is the null vector of W.
- T is linear transformation if and only if $\mathrm{T(cx + y) = cT(x) + T(y)}$.
- If T is linear then $\mathrm{T(x - y) = T(x) - T(y)}$.
- T is linear if and only if $T(\sum_{i=1}^{n} c_i x_i) = \sum_{i=1}^{n} c_i T(x_i)$.

Applications

Applications

- Application of linear transformation is for geometric transformations, such as those performed in computer graphics, where the translation, rotation and scaling of 2D or 3D objects is performed by the use of a transformation matrix.
- Linear transformations also are used as a mechanism for describing change: for example in calculus correspond to derivatives; or in relativity, used as a device to keep track of the local transformations of reference frames.
- Another application is in compiler optimizations of nested-loop code, and in parallelizing compiler techniques.

Examples

Examples

- In calculus, operations of differentiation and integration are two important examples of linear transformation.
- In geometry, Rotations, Reflections, Projections, Dilations are important examples of linear transformation.

Examples

Identity Mapping

Define a map $T : V \to V$ by $T(x) = x$.
Clearly, $T(x + y) = x + y = T(x) + T(y)$
$T(cx) = cx = cT(x)$.
Thus, T is a linear transformation.

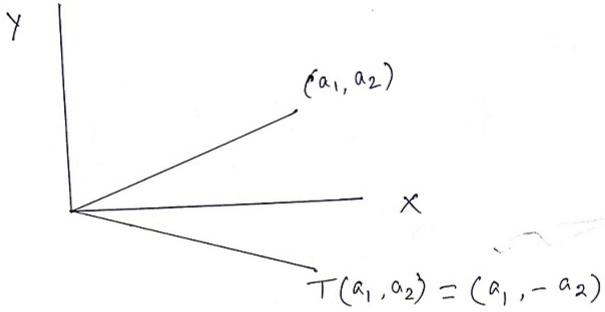
Zero Mapping

Define a map $T : V \to V$ by $T(x) = \theta'$.
Clearly, $T(x+y)=\theta' = \theta' + \theta' = T(x) + T(y)$
$T(cx)=\theta'=c\theta' = cT(x)$.
Thus, T is a linear transformation.

Reflection

Reflection

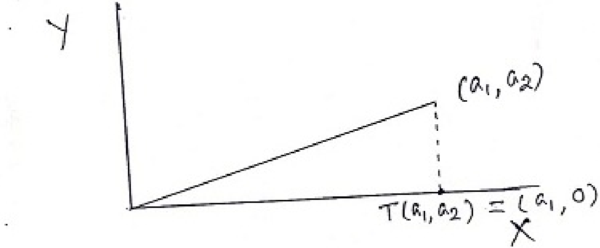
Define, a map $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(a_1, a_2) = (a_1, -a_2)$.
T is the Reflections about X-axis.
Now, $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^{2\times 2}$ then
$T(a+b)$=$(a_1 + b_1, -a_2 - b_2)$
=$(a_1, -a_2) + (b_1, -b_2)$
=$T(a) + T(b)$
Also, $T(ca) = c(a_1, -a_2) = cT(a)$.
Thus, T is a linear transformation.


Y
(a_1, a_2)
X
$T(a_1, a_2) = (a_1, -a_2)$

Projection

Projection

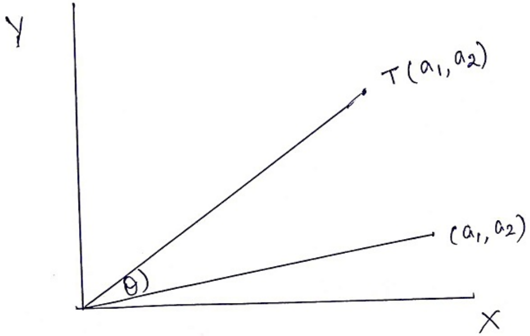
- Define, a map $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(a_1, a_2){=}(a_1, 0)$.
  T is the Projection about X-axis.
  Clearly, T is a linear transformation.
- Generalization of this map $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $T(a_1, a_2. \ldots, a_n){=}(a_1, a_2, \ldots, a_{n-1}, 0)$. T is called the natural Projection.


Y
(a1, a2)
T(a1, a2) = (a1, 0)
X

Rotation

Rotation

Define, a map $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by
$T(a_1, a_2)$=$(a_1 cos\theta - a_2 sin\theta, a_1 sin\theta + a_2 cos\theta)$.
T is the Rotation about angle θ.
Clearly, T is a linear transformation.


Y
T(a1, a2)
(a1, a2)
θ
X

Dilations

Dilations

- Define, a map $T : V \to V$ by $T(v)=av$. T is the Dilations.
- Clearly, T is a linear transformation.
- If a=1 then this mapping is called identity map.
- If a=0 the this mapping is called zero map.

Translation

Translation

Define, a map $T : V \to V$ by $T(x)=x+v$.
T is called the translation.
Clearly, T is not a linear transformation.
T is linear transformation iff v=0.

Examples

Example

Define, a map T: $M_{m \times n} \rightarrow M_{n \times m}$ by $T(A) = A'$.
Clearly, T is a linear transformations.

Example

Define, a map T: $C \rightarrow R$ by $T(x_n)$=lim x_n where C is the set of all convergent real sequence.
Clearly, T is a linear transformations.

Examples

Define, a map T: $C[a,b] \to R$ by $T(f) = \int_a^b f(x)\,dx$ where C[a,b] is the set of all continuous function on [a,b].
Clearly, T is a linear transformations.

Example

Define, a map T: $D[a,b] \to C[a,b]$ by $T(f)$=f' where D[a,b] is the set of all continuously differentiable functions on [a,b].
Clearly, T is a linear transformations.

Kernel of a Linear Transformations

Kernel of a Linear Transformations

Let $T : V \to V$ be a linear transformations. Then Ker T is a subspace of V consisting of all vectors mapped to zero vector of W.
Ker T$= \{x : T(x) = \theta'\}$.
The kernel of a linear transformation measures how close the map is to being the zero map: a linear transformation with a large kernel sends many vectors to zero, while a linear transformation with a small kernel sends few vectors to zero.

Kernel of a Linear Transformations

Ker T is a subspace of V.
Proof: x,y $\in KerT. \Rightarrow T(x) = T(y) = \theta'$
$\Rightarrow T(x + y) = T(x) + T(y) = \theta'$
$\Rightarrow x + y \in KerT.$
Also, cx $\in$ Ker T.
Thus, Ker T is a subspace of V.

Kernel of a Linear Transformations

Kernel of a Linear Transformations

Let $T : V \to V$ be a linear transformations. Then T is injective if and only if Ker T= $\{\theta\}$.

Proof: T is injective and T(θ)=θ'.

$\Rightarrow \theta$ is the only preimage of $\theta' \Rightarrow KerT = \{\theta\}$.

Conversely, Ker T= $\{\theta\}$ Now, T(x)=T(y) $\Rightarrow T(x) - T(y) = \theta'$

$\Rightarrow T(x - y) = \theta'$

$\Rightarrow x - y = \theta$

$\Rightarrow x = y$

Thus, T is injective.

Image of a linear transformations

Image of a linear transformations

Let $T : V \to V$ be a linear transformations. Then Im T is a subset of W consisting of all images of the elements of V.

Im T= $\{T(x) : x \in V\}$.

The image of a linear transformation measures how close the map is to giving all of W as output: a linear transformation with a large image hits most of W, while a linear transformation with a small image misses most of W.

Kernel of a Linear Transformations

Im T is a subspace of V.

Rank and Nullity

Image of a linear transformations

Let $T : V \to V$ be a linear transformations. And Ker T and Im T are finite dimensional.
Rank T= Dimension of Im T
Nullity=Dimension of Ker T.

A linear transformation with a large nullity has a large kernel, which means it sends many elements to zero. Intuitively, larger the nullity, smaller the rank T.
In other words, the more vectors that carried into θ , smaller the range. And vice versa.

Linear Transformations and Bases

A fundamental result is that a linear transformation is completely determined by its values on a basis.

Dimension Theorem

Any linear transformation T : V $\rightarrow$ W is characterized by its values on a basis of V . Conversely, for any basis B $= \{v_i\}$ of V and any vectors $\{w_i\}$, there exists a unique linear transformation T : V $\rightarrow$ W such that $T(v_i) = w_i$ for each $i = 1, 2 \ldots, n$.

Dimension Theorem

Balance between rank and nullity can be understood using the dimension theorem.

Dimension Theorem

Let $T : V \to V$ be a linear transformations. Then
Rank T + Nullity T =dim V

Linear Transformations

Let, V and W be two vector spaces of equal dimension. Let $T : V \to V$ be a linear transformations. Then following results are equivalent:

- T is one-one
- T is onto
- Rank T=dim V

Proof: Rank T + Nullity T =dim V
T is one-one
Ker T= $\{\theta\}$
Rank T=dim V
Also, Rank T=dim V=dim W
dim(Im T)=dim W
Im T=W
T is onto

Linear transformation

Consider the map $T : \mathbb{R}^{3\times1} \to \mathbb{R}^{3\times1}$ by T(x,y,z)=(x+y,y+z,z+x). Prove, that T is one-one and onto.

Ans:(x,y,z) $\in$ Ker T.

T(x,y,z)=(0,0,0)

(x+y,y+z,z+x)=(0,0,0)

x+y=0. y+z=0, z+x=0

x=y=z=0

T is one-one

T is onto.

Isomorphism

Isomorphism

- A linear transformation $T : V \to W$ is called an isomorphism if T is one-to-one and onto. Equivalently, T is an isomorphism if $\ker(T) = 0$ and $\mathrm{im}(T) = W$. We say that two vector spaces are isomorphic if there exists an isomorphism between them.
- Saying that two spaces are isomorphic is a very strong statement, as we will see: it says that the vector spaces V and W have exactly the same structure.
- More specifically, saying that $T : V \to W$ is an isomorphism means that we can use T to relabel the elements of V to have the same names as the elements of W, and that we cannot tell V and W apart at all.

Linear transformation and matrices

Matrices Build Linear Transformations

Suppose that A is an $m \times n$ matrix. Define a function T: $\mathbb{R}^{n\times 1} \to \mathbb{R}^{m\times 1}$ by T(x)= A x. Then T is a linear transformation.

Matrix of a Linear Transformation

Suppose that T: $\mathbb{R}^{n\times 1} \to \mathbb{R}^{m\times 1}$ is a linear transformation. Then there is an $m \times n$ matrix A such that T(x)=Ax

So if we were to restrict our study of linear transformations to those where the domain and codomain are both vector spaces of column vectors, every matrix leads to a linear transformation of this type, while every such linear transformation leads to a matrix. So matrices and linear transformations are fundamentally the same.

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Thank
You